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An application of splittable 4-frames to coloring of $K_{n,n}$

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Dedicated to Alex Rosa on the occasion of his sixty-five birthday

Abstract

Axenovich et al. (J. Combin. Theory Ser. B, to appear) considered the problem of the generalized Ramsey theory. In one case, they use the existence of Steiner triple systems, Pippenger and Spencer's theorem on hyperedge coloring, and the probabilistic method to show that $r'(K_{n,n}, C_4, 3) \leqslant 3n/4(1+o(1))$, where $r'(K_{n,n}, C_4, 3)$ denotes the minimum number of colors to color the edges of $K_{n,n}$ such that every 4-cycle receives at least either 3 colors or 2 alternating colors. In this short paper, using techniques from combinatorial design theory, we prove that $r'(K_{n,n}, C_4, 3) \leqslant (2n/3) + 9$ for all n. The result is the best possible since $r'(K_{n,n}, C_4, 3) > \lfloor 2n/3 \rfloor$ as shown by Axenovich et al. (J. Combin. Theory Ser. B, to appear).

1. Introduction

A *group divisible design* (or GDD) is a triple $(X, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

1. G is a partition of a set X (of *points*) into subsets called *groups*,
2. B is a set of subsets of X (called *blocks*) such that a group and a block contain at most one common point, and
3. every pair of points from distinct groups occurs in a unique block.

The group type of the GDD is the multiset $\{|G|: G \in \mathcal{G}\}$. A GDD $(X, \mathcal{G}, \mathcal{B})$ will be referred to as a K-GDD if $|B| \in K$ for every block B in $\mathcal{B}$.

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A *frame* is a group divisible design $(X,\mathcal{G},\mathcal{B})$ whose block set admits a partition into holey parallel classes, each holey parallel class being a partition of $X\backslash G$ for some $G\in\mathcal{G}$. The groups of a frame are usually referred to as *holes*. The *type* of the frame is the group type of the GDD. A frame of block size k is denoted as k-frame. A frame of type h^u has u holes of size h and is called *uniform*.

It is known [8] that if there is a k-frame of type h^u (with $u>1$), then $u\geqslant h+1$, $h\equiv 0\,(\mathrm{mod}\,k-1)$, and $h(u-1)\equiv 0\,(\mathrm{mod}\,k)$. The necessary conditions for the existence of a uniform k-frame have been proved to be sufficient for $k=3$ [11]. A partial solution for $k=4$ has been proved in [8,4,5].

In this paper, we are interested in a special type of 4-frame. A 4-frame of type $(2g)^u$ is *splittable* if there exist gu points from the frame, g from each group, such that every block of size 4 intersects the gu points in either 1 or 3 points. If a splittable 4-frame of type $(2g)^u$ exists, then there exists a 3-GDD of type g^u.

In this paper, we prove that there exists a splittable 4-frame of type 6^u when $u\equiv 1\,(\mathrm{mod}\,4)$. We also apply a splittable 4-frame of type 6^u to a coloring problem of Axenovich et al. [1] to obtain an asymptotic solution to their problem.

2. Splittable 4-frame

Lemma 2.1. *There exists a splittable* 4-*frame of type* 6^5.

Proof. A 4-frame of type 6^5 is presented in [10]. Let $V=Z_{15}\times\{0,1\}$ and $G=\{\{i_0,i_1,(5+i)_0,(5+i)_1,(10+i)_0,(10+i)_1\}:\ 0\leqslant i\leqslant 4\}$. We start with holey parallel classes. The first holey parallel class is

$$\{9_0,12_0,13_0,1_1\},\quad \{14_0,2_0,3_0,6_1,\},\quad \{4_0,7_0,8_0,11_1\},$$
$$\{9_1,12_1,8_1,11_0\},\quad \{14_1,2_1,13_1,1_0\},\quad \{4_1,7_1,3_1,6_0\}.$$

The second holey parallel class is

$$\{13_0,4_0,6_0,12_1\},\quad \{3_0,9_0,11_0,2_1\},\quad \{8_0,14_0,1_0,7_1\},$$
$$\{13_1,4_1,11_1,2_0\},\quad \{3_1,9_1,1_1,7_0\},\quad \{8_1,14_1,6_1,12_0\}.$$

The remaining 8 classes are obtained by adding 1, 2, 3 and 4, reducing modulo 15. Letting $X=Z_{15}\times\{0\}$, we can see that the 4-frame is splittable since every base block has either one 0 or three 0s in the subscript. □

Lemma 2.2. *There exists a splittable* 4-*frame of type* 6^9.

Proof. Let $V=Z_{54}$ and let the groups be $\{i,9+i,18+i,27+i,36+i,45+i\}$ for $i=0,1,2,\ldots,8$. The blocks are $\{1,2,6,8\}$, $\{3,22,35,47\}$, $\{5,25,33,48\}$, and $\{10,13,34,50\}$ and their translates (mod 54). If we add 18 and 36 to each of 4 blocks, the 12 blocks would cover all elements except the multiple of 9 once. Hence, they form a frame parallel class. The other holey parallel classes can be obtained by adding i to the classes missing the groups $\{0,9,18,27,36,45\}$ for $i=1,2,\ldots,8$. Also, each base block contains either 3 odd or 3 even numbers. When we take $T = 2Z_{27}$, every block intersects T either 1 or 3 times. Hence, the 4-frame is splittable. □

We also note that a skew Room frame construction for a 4-frame [4] also produces a splittable 4-frame. We need a few definitions.

For $1 \leqslant i \leqslant n$, define $H_i = \{x_{j+h(i-1)}\colon\ 0 \leqslant j \leqslant h-1\}$ and let $\mathscr{H} = \{H_i\colon\ 1 \leqslant i \leqslant n\}$; H_i is called a *hole*. A *Room frame of type h^n with hole set $\mathscr{H}$* is a $hn \times hn$ array $\mathscr{F}$, and for each cell of $\mathscr{F}$, indexed by $X = \{x_0, x_1, \ldots, x_{hn-1}\}$,

(a) the cells $(s,t) \in H_i \times H_i$ are empty when $1 \leqslant i \leqslant n$,
(b) each 2-element subset of X that is not a 2-element subset of H_i occurs in exactly one cell of $\mathscr{F}$, and each cell of $\mathscr{F}$ either contains a pair of symbols from X or is empty, and
(c) each row and each column of $\mathscr{F}$ that intersects H_i contains each element from $X \backslash H_i$ exactly once.

A *skew* Room frame is a Room frame in which cell (i,j) is occupied if and only if cell (j,i) is empty.

From a skew Room frame of type h^n defined above, one can get a 4-GDD of type $(6h)^n$ [9]. The 4-GDD is based on $X \times Z_6$ with groups $H_i \times Z_6$, $1 \leqslant i \leqslant n$. The block set $\mathscr{B}$ contains all blocks $\{(a,j),(b,j),(c,1+j),(r,4+j)\}$, where $j \in Z_6$, $\{a,b\} \in \mathscr{F}$, $\{a,b\}$ occurs in column c and row r.

If all quadruples (a,b,c,r) can be partitioned into sets such that each set forms a partition of $X \backslash H_i$ for some i, and each H_i corresponds to $2h$ of the sets, we call the skew Room frame *partitionable* [4]. It is clear that from a partitionable skew Room frame, each set which partitions $X \backslash H_i$ will result in a holey parallel class in the resulting 4-GDD.

Lemma 2.3 (Colbourn et al. [4]). *If there exists a partitionable skew Room frame of type h^r, then there exists a* 4*-frame of type* $(6h)^n$.

Now, we note that each block has the form $\{(a,j),(b,j),(c,1+j),(r,4+j)\}$ for $j \in Z_6$ which has either 3 odd numbers or 3 even numbers in the second component (mod 6). Hence, if we let $T = X \times \{0,2,4\}$, every block in the 4-GDD of type $(6h)^u$ intersects T in either 1 or 3 points. Therefore, we have the following lemma:

Lemma 2.4. *If there is a partitionable skew Room frame of type h^n, then there exists a splittable* 4*-frame of type* $(6h)^n$.

Lemma 2.5. *There exists a splittable* 4*-frame of type* 6^u *when* $u = 13, 17, 29$.

Proof. A partitionable skew Room frame of type 1^u for $u = 13, 17, 29$ is constructed in [4]. □

Lemma 2.6. *There exists a splittable* 4*-frame of type* 6^{33}.

Proof. A partitionable skew Room frame of type 4^8 is constructed in [4]. This implies the existence of a splittable 4-frame of type 24^8. Adding six infinite points, and filling each group of size 24 together with the six infinite points using a splittable 4-frame of type 6^5 implies the existence of a splittable 4-frame of type 6^{33}. □

The following recursive construction can be easily seen:

Lemma 2.7. *If there exists a K-GDD of type* 1^v, *and for every* $k \in K$ *there exists a splittable* 4*-frame of type* $(2g)^k$, *then there exists a splittable* 4*-frame of type* $(2g)^v$.

We use a PBD result cited in [2].

Theorem 2.8. *If* $v \equiv 1 \pmod 4$, *there exists a* $\{5, 9, 13, 17, 29, 33\}$*-GDD of type* 1^v.

Theorem 2.9. *If* $u \equiv 1 \pmod 4$, *then there exists a splittable* 4*-frame of type* 6^u.

3. On certain coloring of $K_{n,n}$

Given graphs G and H, a coloring of $E(G)$ is called (H, q)-coloring if the edges of every copy of $H \subset G$ together receive at least q colors. Let $r(G, H, q)$ denote the minimum number of colors in an (H, q)-coloring of G.

Axenovich et al. [1] proved the following:

Theorem 3.1. *If n is odd*, *then* $r(K_{n,n}, C_4, 3) \leqslant n$. *If n is even*, *then* $r(K_{n,n}, C_4, 3) \leqslant n+1$.

It is worth noting that the existence of an N_2-Latin square can improve the result slightly when n is even [6].

Also, Axenovich, Füredi, and Mubayi proved the following lower bound on $r(K_{n,n}, C_4, 3)$:

Theorem 3.2. $r(K_{n,n}, C_4, 3) > \lfloor 2n/3 \rfloor$.

In proving the lower bound, they did not use the fact that the C_4 can contain an alternating C_4, a 2-colored C_4 whose edges alternate between its two colors when viewed cyclically.

Hence, they defined the following [1]. A *weak* $(C_4, 3)$-coloring of $K_{n,n}$ is a coloring of the edges of $K_{n,n}$ in which every copy of C_4 has at least three colors or is alternately 2-colored. Let $r'(K_{n,n}, C_4, 3)$ denote the minimum number of colors in a weak $(C_4, 3)$-coloring of $K_{n,n}$. Using a sophisticated theorem of Pippenger and Spencer [7] and the probabilistic method, they proved the following:

Theorem 3.3. *As* $n \to \infty$, $r'(K_{n,n}, C_4, 3) \leqslant 3n/4(1 + o(1))$.

We will use the splittable 4-frame of type 6^u to improve their result.

Theorem 3.4. *If* $r'(K_{g,g}, C_4, 3) \leqslant r_g$ *and there exists a partitionable* 4*-frame of type* $(2g)^u$, *then* $r'(K_{gu,gu}, C_4, 3) \leqslant (gu/3) + r_g - g/3$.

Proof. Let $(X, \mathscr{G}, \mathscr{B})$ be a partitionable 4-frame of type $(2g)^u$ and V be the set of size gu such that every block intersects T in either 1 or 3 points. We construct a coloring on the complete bipartite graph with parts T and $X \setminus T$. Every block intersects T in 1 or 3 points; therefore, every block induces 3 edges that has one vertex in T and the other in $X \setminus T$. We first outline the idea of the construction. We will construct a color class for every frame parallel class, together with a color class in a $K_{g,g}$ in which the frame parallel class is missing. So, it will use up $g/3$ color classes from the u $K_{g,g}$s (from the u holes). Hence, we need to build $r_g - g/3$ color classes for the remaining edges inside the u $K_{g,g}$s. The actual construction is as follows. For every frame parallel class that misses G_i for some i, we color the 3 crossing edges from every block in the frame parallel class with the same color. Also, because $|T \cap G_i| = |(X \setminus T) \cap G_i| = g$, we put a color class of the $K_{g,g}$ with parts $(T \cap G_i)$ and $(X \setminus T) \cap G_i$. Since there are $g/3$ frame parallel classes missing G_i, we have only used $g/3$ color classes in the $K_{g,g}$ on parts $(T \cap G_i)$ and $(X \setminus T) \cap G_i$. Therefore, there are $r_g - g/3$ color classes in the $K_{g,g}$, with parts $(T \cap G_i)$ and $(X \setminus T) \cap G_i$ for every i. We build each of the $r_g - g/3$ new color classes by taking the edges of one remaining color class in $K_{g,g}$ with parts $(T \cap G_i)$ and $(X \setminus T) \cap G_i$ for all i.

Next, we show that every edge is colored exactly once for any pair of edges $\{a, b\}$ where $a \in T$ and $b \notin T$. Then, either $\{a, b\} \subset G_i$ for some i or $\{a, b\} \not\subset G_i$ for any i. In the former case, the edge $\{a, b\}$ is colored exactly once because all edges of $K_{g,g}$ are colored once. In the latter case, $\{a, b\}$ is contained in a unique block in the 4-frame. Hence, it is also colored.

Finally, we show that any C_4 has either 3 colors or 2 colors, in which case it must be alternately colored. From the proof of the lower bound in [1], these conditions are equivalent to show that:

1. each connected component of the subgraph induced by any color class is a star, and
2. any 2 points in the same partite can be the endpoint of at most one monochromatic paths of length 2.

The first condition is easily satisfied since every block in the frame parallel class is disjoint. The second condition is also satisfied. If there are 2 monochromatic paths of length 2 with common endpoints, then the 2 points cannot be in the same G_i because it contradicts the definition of $r'(K_{n,n}, C_4, 3)$; if the points are in a different group, then they are in 2 blocks of size four in the frame, which is clearly impossible. □

Since $r'(K_{3,3}, C_4, 3) = 3$, we have the following corollary:

Corollary 3.5. *If there exists a splittable* 4*-frame of type* 6^n, *then* $r'(K_{3n,3n}, C_4, 3) = 2n + 1$.

Proof. The lower bound $r'(K_{3n,3n}, C_4, 3) > 2n$ follows from Theorem 3.2. □

Corollary 3.6. $r'(K_{n,n}, C_4, 3) \leqslant \lfloor 2n/3 \rfloor + 9$.

Proof. It is proved in Lemma 2.9 that when $u \geqslant 1$, there exists a splittable 4-frame of type 6^{4u+1}. Hence, we have $r'(K_{12u+3,12u+3}, C_4, 3) = 8u + 3$ by Corollary 3.5. For any n, let m be the smallest integer such that $m \geqslant n$ and $m \equiv 3 \pmod{12}$. If we start with a coloring on $K_{m,m}$ and delete $m - n$ points from each group, it will induce a coloring on $K_{n,n}$ with at most $(2m/3) + 1 < (2n/3) + 9$ colors for all n. □

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References

[1] M. Axenovich, Z. Füredi, D. Mubayi, On generalized Ramsey theory: the bipartite case, J. Combin. Theory Ser. B, to appear.

[2] F.E. Bennett, H.-D.O.F. Gronau, A.C.H. Ling, R.C. Mullin, PBD-closure, in: C.J. Colbourn, J.H. Dinitz (Eds.), CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 1996, pp. 203–213.

[3] T. Beth, D. Jungnickel, H. Lenz, Design Theory, 2nd Edition, Cambridge University Press, Cambridge, 1999.

[4] C.J. Colbourn, D.R. Stinson, L. Zhu, More frames with block size four, J. Combin. Math. Combin. Comput. 23 (1997) 3–19.

[5] G. Ge, Uniform frames with block size four and index one or three, preprint, 1999.

[6] A.C.H. Ling, Some applications of combinatorial design theory to extremal graph theory, Ars Combin. to appear.

[7] N. Pippenger, J. Spencer, Asymptotic behavior of the chromatic index for hypergraph, J. Combin. Theory Ser. A 51 (1989) 24–42.

[8] R. Rees, D.R. Stinson, Frames with block size four, Canad. J. Math. 44 (1992) 1030–1049.

[9] C.A. Rodger, Linear spaces with many small lines, Discrete Math. 129 (1994) 167–180.

[10] D.R. Stinson, The equivalence of certain incomplete transversal designs and frames, Ars Combin. 22 (1986) 81–87.

[11] D.R. Stinson, Frames for Kirkman triple systems, Discrete Math. 65 (1987) 289–300.

[12] R.M. Wilson, An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures, J. Combin. Theory Ser. A 13 (1972) 246–273.